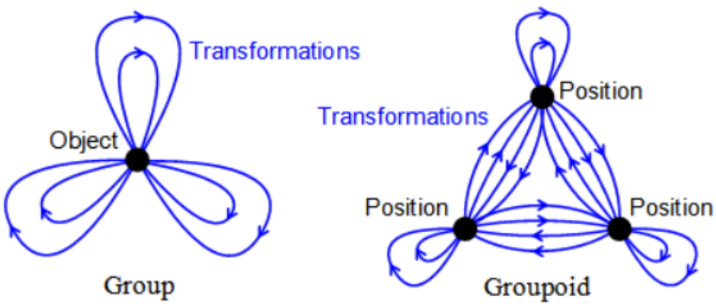
WHAT IS A GROUPOID?

OWEN TANNER

1. Introduction

Definition 1.1. *(Abstract) A groupoid is a small category of isomorphisms.*



This means a group is a groupoid. Let's think of it a bit like a group then.

Definition 1.2. *(Concrete)*
A groupoid is a set $\mathcal{G}$ with partially defined multiplication $\gamma_1\gamma_2$ and everywhere defined involutive operation $\gamma \mapsto \gamma^{-1}$, satisfying:

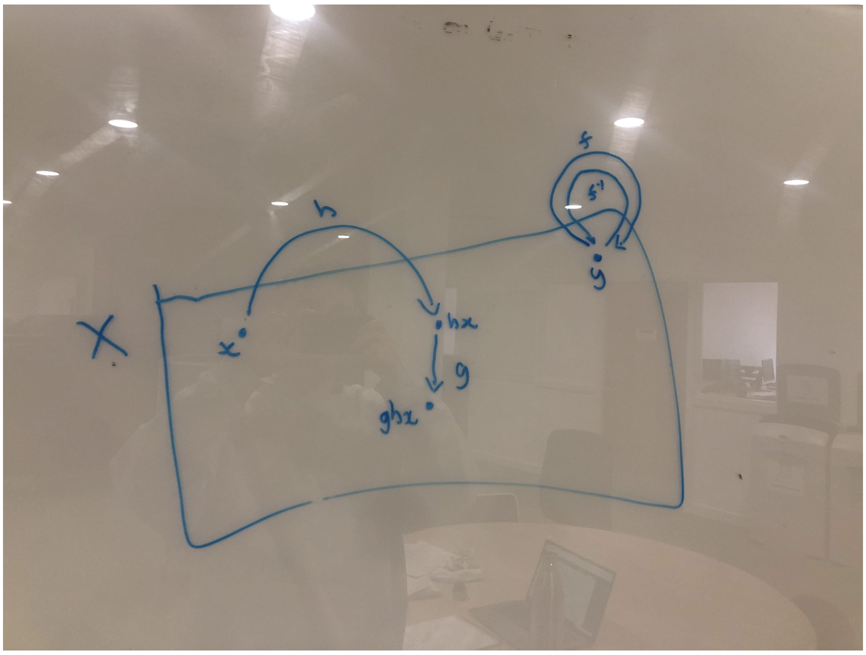
(1) *Associativity: If $\gamma_1\gamma_2$ and $(\gamma_1\gamma_2)\gamma_3$ are defined, then $\gamma_2\gamma_3$ is defined and $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$*
(2) *Existence of r, s: The range and source maps $r(\gamma) = \gamma\gamma^{-1}$, $s(\gamma) = \gamma^{-1}\gamma$ are always well defined. If $\gamma_1\gamma_2$ are well defined, then $\gamma_1 = \gamma_1\gamma_2\gamma_2^{-1}$, $\gamma_2 = \gamma_1^{-1}\gamma_1\gamma_2$.*

Definition 1.3. *Heuristic A groupoid is a generalisation of a group action. So we think of it like a generalised dynamical system.*

Notation: elements of the form $\gamma\gamma^{-1}$ are called units and the space of units is denoted $\mathcal{G}^{(0)}$. The space of composable pairs is denoted by $\mathcal{G}^{(2)}$. We now give a key example- a *transformation groupoid.*

Example 1.4. *Let $\Gamma \curvearrowright X$ be a group acting by homeomorphisms on a topological space X. Then we define the transformation groupoid $\Gamma \ltimes X$ to be the set of pairs $(g, x) \in \Gamma \times X$ here composable pairs are of the form $(g, h(x))(h, x)$ and composition is given by $(g, h(x))(h, x) = (gh, x)$. Here $s(g, x) = (1, x)$, $r(g, x) = (1, g(x))$ and the unit space is canonically identified with X.*

Date: April 7, 2022.



Note that we have many properties of actions translate into properties of the transformation groupoid:

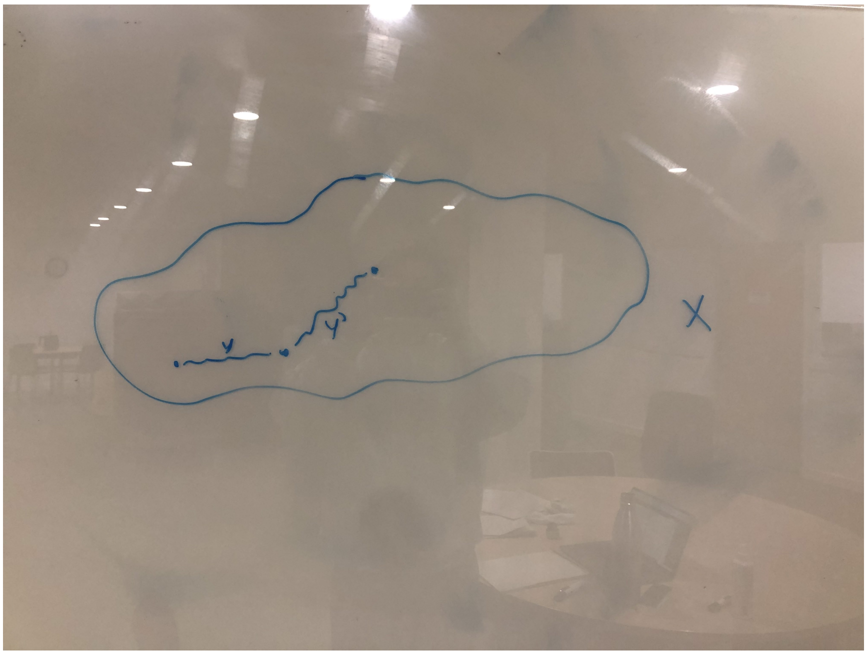
- *So we think of groupoids as "acting on their unit space" in general.*
- *Notice that this means both an arbitrary topological space and an arbitrary group are examples of groupoids. This is by taking the trivial action $1 \curvearrowright X$ on an arbitrary space X, or by taking the trivial action $G \curvearrowright \star$ on the one-point space.*
- *Many algebraic properties of such a groupoid correspond to analytic or geometric properties of an action.*

Just like a rubiks cube is a good example of a group, the 15 puzzle is a good example of a groupoid.

Example 1.5 (15 Puzzle)**.** *15puzzle.netlify.app*

This is because we can't always compose moves, unlike a rubiks cube. This also means there is fundamental differences here. Whilst god's number for a rubiks cube is 20, it doesn't exist for a 15 puzzle. Another example would be the minus cube (popular russian game).

Example 1.6 (Path space)**.** *Another good example is the fundamental groupoid of a topological space X. Consider the space of continious paths in X under homotopy equivalence. We see paths f, f' are composable iff $f(1) = f'(0)$. f^{-1} Also, $s(f) = Id_{f(0)}$, $r(f) = Id_{f(1)}$. The map $\phi : \mathcal{G}^0 \to X \quad Id_x \mapsto x$ identifies the unit space with X. f^{-1} is going to be the path given by $t \mapsto f(1-t)$.*



Example 1.7. *Disjoint union of groups. Let us take G_1, G_2 to be groups. Then there disjoint union $G_1 \cup G_2$ is a groupoid, where we can only compose elements from the same group.*

Definition 1.8. *(Isotropy Group) The isotropy group of a groupoid at some unit $u \in \mathcal{G}^0$ is the group of all elements $g \in \mathcal{G}$ such that $r(g) = s(g) = u$. This is a group because all compositions are well defined in this subspace (sufficient to be a group).*

This is an important notion: note that the isotropy groups of the fundamental groupoid give the fundamental group of a topological space. Otherwise, the isotropy groups of a disjoint union of groups is those groups.

Example 1.9. *(Equivalence relation groupoid) Let X be a set with an equivalence relation $\sim$. We can make a groupoid from this:*

- *Objects are elements of X*
- *For any $x, y \in X$, there is a single morphism (x, y) iff $x \sim y$.*
- $(x, y)(y, z) = (x, z)$.

You are probably wondering why this has deep relevance to C*-algebras and so many people that are interested in C*-algebras also care about groupoids. This is partly cultural– really canonical examples from C* algebras seem to come from groupoids. In fact, my supervisor Prof. Xin Li made such a statement precise back in 2020:

Theorem 1.10 (Li, 2020)**.** *Every classifiable C^* algebra comes from a (might need to be twisted?!) groupoid.*

It's also because the functor $\mathcal{G} \to C_r^*(\mathcal{G})$ oftentimes helps us compute the K - Theory, which is what we use to classify C^*- algebras.