A GENTLE INTRODUCTION TO TOPOLOGICAL FULL GROUPS (GOTHENBURG GROUPOIDS SEMINAR)

OWEN TANNER

1. Outline

This talk and these notes follow a what-why-how format. First I will tell you *what* a topological full group is. Then, I will tell you a bit about *why* people are interested in topological full groups through some very strong black box results. Finally, and for the majority of the talk I will describe *how* one works with topological full groups in practice, by walking everyone through a particularly illuminating and beautiful example.

2. What is a topological full group?

Topological full groups were first introduced for Cantor minimal systems, and the first definition in terms of orbit cocycles. Today I'm presenting a much more general definition, though not as general case as the state of the art which is Nyland-Ortega's definition for locally compact ample groupoids.

Definition 2.1. *(Standing assumptions on the groupoid). Let $\mathcal{G}$ be a étale groupoid with unit space homeomorphic to the Cantor space X. We call such a groupoid a Cantor groupoid.*

As seen in the construction of groupoid C*-algebras, a lot of information about groupoids can be seen from looking at subsets called bisections:

Definition 2.2 (Inverse Semigroup of Open Compact Bisections)**.** *An open subset $B \subset \mathcal{G}$ is called an open bisection if*

$$s : B \mapsto s(B) \subset \mathcal{G}^0 \quad b \mapsto b^{-1}b$$

$$r : B \mapsto r(B) \subset \mathcal{G}^0 \quad b \mapsto bb^{-1}$$

Are homeomorphisms. Let the set of open compact bisections be denoted $\mathcal{B}^{o,k}_{\mathcal{G}}$ Recall that one can multiply subsets of groupoids:

$$B \cdot \hat{B} = \{\gamma \cdot \hat{\gamma} \;:\; (\gamma, \hat{\gamma}) \in \mathcal{G}^{(2)} \cap B \times \hat{B}\}$$

The set of compact open bisections forms an inverse semigroup which is acting on the unit space by partial homeomorphisms.

$$\alpha_B = (r_{_B}) \circ (s_{_B})^{-1} : s(B) \to r(B)$$

A groupoid is étale iff the set of open bisections forms a subbase for the topology on $\mathcal{G}$. In this case the topology is encoded inside the inverse semigroup. In some sense, we can think of groupoids as inverse semigroups acting on spaces and vice versa.

The following definition is due to Nekrashevych, where we require the source and range to be the whole of the unit space:

Date: May 27, 2022.

Definition 2.3. *(Topological Full Group)*

$$[[\mathcal{G}]] = \{B \in \mathcal{B}_{\mathcal{G}}^{o,k} \; : \; s(B) = r(B) = \mathcal{G}^0\}$$

Forms a group with respect to the above multiplication.

Note that if we look at our local homeomorphism it has now magically become a **full homeomorphism**:

$$\alpha_B = (r_B) \circ (s_B)^{-1} : \mathcal{G}^0 \to \mathcal{G}^0$$

3. Why are we interested in topological full groups?

2 main reasons:

(1) They are good invariants for dynamical systems.
  - Complete invariant of flip conjugacy for Cantor minimal systems. Giordano-Putnam-Skau
  - Complete invariant of conjugacy for effective minimal Hausdorff Cantor groupoids. Matui

(2) They provide a way to generate and analyse very interesting groups. In particular, infinite groups that are finitely generated or presented.
  - Monod-Juschenko: There exist *infinite*, finitely generated simple groups which are amenable. (First Example!)
  - Nekrashevych/Matui: The Higman-Thompson groups are the topological full groups of SFT groupoids. Example I hope to get through.
  - Many algebraic and analytic properties of the groups are then translated into (usually much simpler) topological properties of the groupoid. For example there are results concerning finite generation, finite presentation, the Haagerup property, existence of simple subgroups.
  - Matui: We get a free unitary representation inside the reduced groupoid C* -algebra. Full bisections induce unitary elements in the C* algebra. If asked for something precise here, it is given by:

    $$[[\mathcal{G}]] \cong \frac{\{v \in U(C_r^*(\mathcal{G})) \; : \; vC_0(\mathcal{G}^0)v^* = C_0(\mathcal{G}^0)\}}{U(C_0(\mathcal{G}^0))}$$

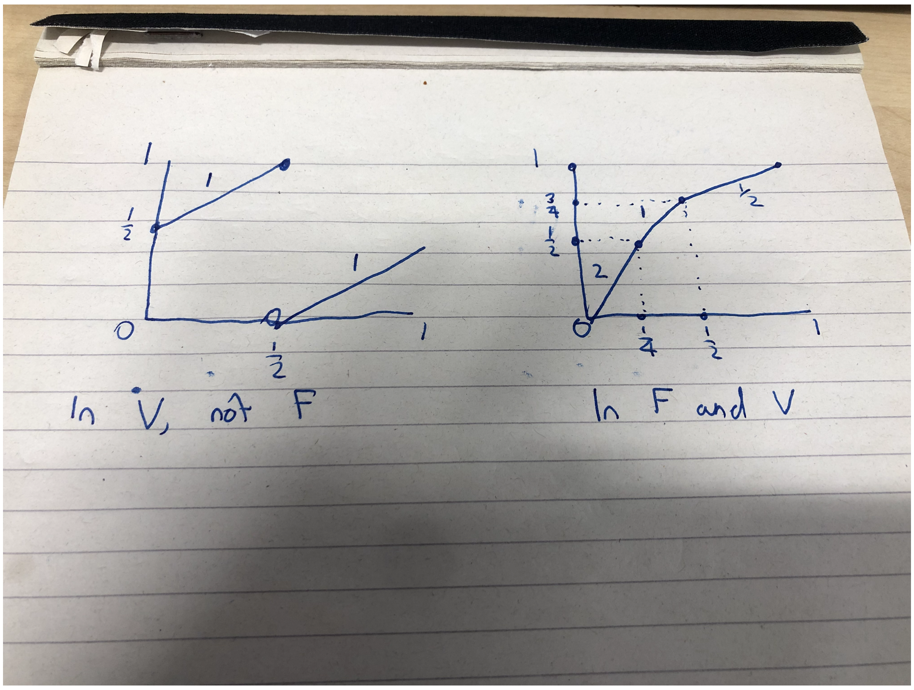
    Where $U(A)$ is the unitary subgroup of A.

4. How can one see a connection between the Cuntz algebra $\mathcal{O}_2$ and Thompsons group V_2

Definition 4.1 (V_2)**.** *V_2 is the set of right continious, piecewise linear bijections of $[0,1]$ with:*

- *Finitely many nondifferentiable points, all of which in $\mathbb{Z}[1/2] := \{\frac{a}{2^n} \quad : a, n \in \mathbb{Z}\}$.*
- *Slopes in $\langle 2 \rangle = \{2^n \; : \; n \in \mathbb{Z}\}$.*

This group is interesting for all kinds of reasons, it is finitely generated and finitely presented. If you consider the subgroup F of *continious* piecewise linear bijections of $[0,1]$ this is a possible counterexample to the von Neumann conjecture (nonamenable but doesn't contain the free group F_2).



Let the Cuntz algebra $\mathcal{O}_2$ be generated by S_0, S_1. If μ be some finite word in 0,1 then the element S_μ is just the product of the corresponding $S_i's$ Nekrashevych observed: $V_2 \cong \{S \in U(\mathcal{O}_2) \, : \, S = \sum_{i=1}^n S_{\mu_i} S_{\nu_i}^*\}$

Not going to explain exactly how this map works, but it is encoded in binary expansions and partitions into them, for example, the element in V not F would swap the first digit in the binary expansion from 0 to 1 and vice versa. So it is associated to $S_1 S_0^* + S_0 S_1^*$.

Note that we also can assume that the length of each interval is 2^{-n} for some n positive. This is because, say we have some interval $[a, c)$ where $a - c \neq 2^{-n}$ for some n. Then, $a - c$ has some finite binary expansion:

$$a - c = \sum_{i=1}^{N} \epsilon_i 2^{-i}$$

So

$$[a, c) = \sqcup_{l=1}^{N-1} [a + \sum_{i=1}^{l} \epsilon_i 2^{-i}, a + \sum_{i=1}^{l+1} \epsilon_i 2^{-i})$$

. **Question: Why does this group appear?!?!**

Answer: Topological Full Groups of Groupoids!

The canonical groupoid model for $\mathcal{O}_2$. Let $X = \{0,1\}^{\mathbb{N}}$ be cantor space. Consider the DR groupoid; for $k \in \mathbb{Z}$ we say

$$(x_n)_n \sim_k (y_n)_n \iff \exists N \in \mathbb{N} \; s.t. \; n > N \implies x_n = y_{n+k}.$$

Then, the groupoid elements are given by:

$$((x_n)_n, k, (y_n)_n)) \quad (x_n)_n \sim_k (y_n)_n$$

Composable pairs are of the form:

$$((x_n)_n, k, (y_n)_n))((\hat{y}_n)_n, \hat{k}, (z_n)_n)) \; s.t. \; (y_n)_n \sim_0 (\hat{y}_n)_n$$

Inverses are given by:

$$((x_n)_n, k, (y_n)_n))^{-1} = ((y_n)_n, -k, (x_n)_n))$$

Hence,

$$s((x_n)_n, k, (y_n)_n)) = ((x_n)_n, 0, (x_n)_n), \; r((x_n)_n, k, (y_n)_n)) = ((y_n)_n, 0, (y_n)_n).$$

A basis for the topology on $\{0,1\}^{\mathbb{N}}$ is given by:

$$C_\mu = \mu + \{0,1\}^{\mathbb{N}}$$

Where μ is some finite word- i.e. arbitrary sequences that start with μ. Then, a basis for the topology on $\mathcal{G}$ is given by:

$$C_{\mu,\nu} = \{((x_n)_n, k, (y_n)_n)) \,:\, (x_n)_n \in C_\mu, \; (y_n)_n \in C_\nu, \; k = |\nu| - |\mu|\}$$

This groupoid is étale and the above are the open bisections. In fact these sets above are clopen and therefore compact. All open compact bisections are disjoint unions of these elements. It is not hard to see then that since $s(C_{\mu,\nu}) = C_\mu$ $r(C_{\mu,\nu}) = C_\nu$ elements of our topological full group are of the following form:

$$B = \sqcup_{i=1}^N C_{\mu_i,\nu_i} \text{ s.t. } X = \sqcup_i C_{\mu_i} = \sqcup_i C_{\nu_i}$$

i.e. so that every sequence starts with exactly one μ_i and exactly one ν_i. Now let us notice something subtle. Any finite word in $\{0,1\}$ corresponds uniquely to a dyadic number in $(0,1)$ via it's binary expansion, for an explicit example:

$$0110 \mapsto 0/2 + 1/4 + 1/8 + 0/16 = 3/8$$

This lets us define a map:

$$f : \cup_N \{0,1\}^N \to \mathbb{Z}[1/2] \cap (0,1)$$

Furthermore, since every number has an infinite binary expansion, we can extend such a map to a map $\{0,1\}^{\mathbb{N}} \to (0,1)$. It is in this way that we see such an identification:

$$f(C_\mu) = [f(\mu), f(\mu) + 2^{-|\mu|}].$$

So that each $C_{\mu,\nu}$ gives us a pair of intervals with length in $\langle 2 \rangle$ and end points in $\mathbb{Z}[1/2] \cap [0,1]$. I think now we have motivated the following map:

$$\Phi : \mathcal{B}_{\mathcal{G}}^{o,k} \to \{\text{partial homeomorphisms of } (0,1) \text{ by linear maps }\} \quad C_{\mu,\nu} \to f_{\mu,\nu}$$

Where

$$f_{\mu,\nu} : [f(\mu), f(\mu) + 2^{-|\mu|}) \to [f(\nu), f(\nu) + 2^{-|\nu|}) \quad t \mapsto f(\nu) - f(\mu) + 2^{|\mu|-|\nu|} t$$

Considering Φ on the TFG we get therefore an isomorphism:

$$\Phi : [[\mathcal{G}]] \to V_2$$

It is easy to see that $ker(\Phi) = \mathcal{G}^0 = (X, 0, X)$. To see surjectivity, consider an arbitrary $f \in V_2$. Since f is bijective, it is enough for each piecewise linear component to find a corresponding cylinder set $C_{\mu,\nu}$. Let

$$f_i : [a, a + 2^{-k}) \to [c, c + 2^{-m}) \quad t \mapsto c + 2^{k-m}(t - a)$$

be some piecewise linear component of f. We may assume wlog that $2^k a, 2^m c \in \mathbb{N}$ since otherwise, we can just rewrite f_i as the union of 2^n functions of the form:

$$f_l : [a + l2^{-n}, a + l2^{-(n+k)}) \to [c + l2^{-n}, c + l2^{-(n+m)})) \quad t \mapsto c + 2^{k-m}(t - a)) \quad l \in \{0, 1, 2, 3..., 2^n\}$$

For suitably large n.

Then let μ be the binary expansion of $2^k a$, ν be the binary expansion of $2^m c$. We see that $f_i = \Phi(C_{\mu,\nu})$ so that we are done.
Note that moving to C* algebras, in some sense the characteristic function on $C_{\mu,\nu}$ give us $S_\mu S_\nu^*$, since $(\chi_{C_i} \mapsto S_i)$. Therefore, we get exactly the deep reason behind Nekrashevych's observation.